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Heat transfer from a sphere at low Reynolds numbers

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The heat transfer due to forced convection from an isothermal sphere in a steady stream of viscous incompressible fluid is calculated for low values of the Reynolds number and Prandtl numbers of $O(1)$. The mean Nusselt number is compared with the results of experimental measurements. At very low Reynolds numbers, both the local and mean Nusselt numbers are compared with the results obtained from the theory of matched asymptotic expansions.

1. Introduction

In this paper the problem of heat transfer from an isothermal sphere in a steady flow at small Reynolds number is considered. The physical properties of the fluid are assumed to be independent of temperature and, in particular, the fluid is assumed to be incompressible. The problem of heat transfer is in general a complex one involving four independent parameters see (Schlichting 1960, p. 297). When free convection is taking place the fluid motion is altered by the effect of gravitational forces on heated fluid particles of variable density; in addition, heat is generated internally in the fluid by viscous dissipation. In the present paper these latter processes are assumed negligible and the problem of forced convection only is considered. This imposes the restrictions on the problem that the normalized temperature difference $|T_s - T_\infty|/T_\infty$, where T_s is the constant temperature of the sphere and T_∞ is the temperature of the fluid at large distances from the sphere, must be small in comparison with unity and further that it must be large compared with the square of the Mach number of the basic flow.

The basic theory of this problem has been described by Illingworth (1963), but very little theoretical work exists. Part of the difficulty is that theoretical solutions for the velocity of the fluid must be available in order to consider the forced-convection problem. Acrivos & Taylor (1962) assumed that the velocity field was given by the Stokes solution for slow flow past a sphere and used the method of matched inner and outer expansions to obtain an expression for the mean Nusselt number N_m as a function of the Péclet number $Pe = \sigma R$. Here σ is

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the Prandtl number and R the Reynolds number. This work was extended by Rimmer (1968), with subsequent minor corrections (Rimmer 1969), by using the expression for the velocity field given by Proudman & Pearson (1957). This is valid for larger values of R than the Stokes expression for the velocity and thus the expression for N_m given by Rimmer (1968) is valid for larger R than that of Acrivos & Taylor (1962). Both expressions agree up to terms in σ^2R^2, but thereafter the expansion of Rimmer (1968) ceases to depend on σR alone and is a function of σ and R separately. The theory could be further extended by using the velocity distributions given by Chester & Breach (1969). One other study of interest in this context is that of Kassoy, Adamson & Messiter (1966), who considered the modifications in this type of approach for compressible low Reynolds number flow in which significant variations in density occur.

The range of validity of these theoretical expansions is not known, except that R must be small and σ at most of $O(1)$. It therefore seems worthwhile to calculate the heat transfer numerically for a range of small values of the Reynolds number and various values of the Prandtl number. The calculations are described in the present paper. They have been carried out by two methods, in both cases using velocity distributions calculated from the exact Navier–Stokes equations. The first method employs velocity fields calculated by Dennis & Walker (1971) for the range $R = 0{\cdot}1$–40, where R is the Reynolds number based on the diameter of the sphere, using the semi-analytical method of series truncation. Here the stream function and vorticity were expressed in terms of series of Legendre functions with argument $z = \cos\theta$, where (r, θ) are polar co-ordinates in a plane through the axis of symmetry of the motion. The coefficients of the Legendre functions in the series are functions of the variable $\xi = \log(r/a)$, where a is the radius of the sphere, and these were found as numerical solutions of sets of ordinary differential equations which express the fact that the Navier–Stokes equations for steady incompressible viscous flow past the sphere shall be satisfied. A similar method of series truncation is employed in the present paper to solve the forced-convection problem. Here the fluid temperature is expressed as a series of Legendre polynomials with argument $z = \cos\theta$ and coefficients which are functions of ξ.

The basis of the series truncation method is to approximate the solution, which in theory consists of an infinite series, by a finite number of terms. In the present case, if R is small and σ of $O(1)$ only relatively few terms are required to give an adequate approximation to the temperature field. In these circumstances this method is felt to be superior to a conventional finite-difference approach (see Dennis & Walker 1971, p. 787). If either R or σ is increased, the complexity of the temperature field is increased, more terms of the series are required and the effectiveness of the method diminishes. Generally the number of terms required to represent adequately a particular solution is not known and must be obtained by experience. It is therefore desirable to check the results in at least one case by an independent method. For $\sigma = 0{\cdot}73$ (air), independent results are obtained by solving the energy equation for the temperature using two-dimensional finite-difference methods. An independent velocity field is used for this calculation. This is obtained by solving the Navier–Stokes equations numerically over the range $R = 0{\cdot}5$–20 using two-dimensional finite-difference methods. The agreement between the two sets of results is extremely good. Results are obtained using the series truncation method for the range $R = 0{\cdot}1$–10 for the values $\sigma = 0{\cdot}73$ and 1 and for $R = 0{\cdot}1$–1 for $\sigma = 8$.

The local and mean Nusselt numbers are calculated from the temperature distributions. For very low Reynolds numbers and for Prandtl numbers of order unity these quantities are in very good agreement with the theory of Rimmer (1968). The mean Nusselt number is compared with various formulae based on experimental results. The agreement for very low Reynolds numbers is only fair, but it improves substantially as the Reynolds number is increased. The experimental correlation which fits the present results best is that given by Whitaker (1972).

2. Basic equations

A spherical polar co-ordinate system (r, θ, ϕ) with origin at the centre of the sphere is chosen with $\theta = 0$ as the downstream radius. Both the fluid motion and temperature field are axially symmetric and hence independent of the azimuthal co-ordinate ϕ. The fluid motion is described by radial and transverse components of velocity (u, v) in a plane through the axis of symmetry. These are obtained by dividing the corresponding dimensional components by the main-stream velocity U_∞. The velocity components are expressed in terms of a dimensionless stream function $\psi(\xi, \theta)$ by the equations

$$u = \frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\xi}. \tag{1}$$

In a recent paper, Dennis & Walker (1971) have shown that if we write

$$\psi = e^{\frac{1}{2}\xi}\sum_{n=1}^{\infty} f_n(\xi)\int_z^1 P_n(t)\,dt, \tag{2}$$

where the $P_n(z)$ are the ordinary Legendre polynomials, numerical solutions for the functions $f_n(\xi)$ may be obtained by solving sets of ordinary differential equations.

If the physical properties of the fluid are assumed constant and the internal generation of heat by friction is neglected, the energy equation is then

$$\frac{\partial^2 T}{\partial\xi^2} + \frac{\partial T}{\partial\xi} + \cot\theta\frac{\partial T}{\partial\theta} + \frac{\partial^2 T}{\partial\theta^2} = \frac{R\sigma}{2}e^{\xi}\left(u\frac{\partial T}{\partial\xi} + v\frac{\partial T}{\partial\theta}\right). \tag{3}$$

Here $T(\xi, \theta)$ is the normalized temperature difference, obtained by subtracting the main-flow temperature T_∞ from the temperature and dividing by $T_s - T_\infty$. The boundary conditions are then

$$T(0, \theta) = 1, \quad T \to 0 \quad \text{as} \quad \xi \to \infty. \tag{4a, b}$$

Here $R = 2U_\infty a/\nu$ is the Reynolds number based on the diameter (twice that used by Rimmer).

The present approach is to write $T(\xi, \theta)$ as

$$T(\xi, \theta) = \sum_{n=0}^{\infty} t_n(\xi)\,P_n(z), \tag{5}$$

which satisfies the requirement that $\partial T/\partial\theta = 0$ on $\theta = 0, \pi$. Then in accordance with (4) the t_n satisfy

$$t_n(0) = \delta_{n0}, \qquad t_n(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \tag{6a, b}$$

where δ_{nj} is the Kronecker delta. Substitution of the series (5) into the basic equation (3) leads to the infinite set of equations

$$t_n'' + (1 - A_n)t_n' - \{n(n+1) + B_n\}t_n = S_n, \tag{7}$$

where the prime denotes differentiation with respect to ξ and

$$A_n(\xi) = \tfrac{1}{2}R\sigma\, e^{-\frac{1}{2}\xi} \sum_{k=1}^{\infty} \alpha_{2k,n}^{n} f_{2k}, \tag{8}$$

$$B_n(\xi) = \tfrac{1}{2}R\sigma\, e^{-\frac{1}{2}\xi} \sum_{k=1}^{\infty} \beta_{2k,n}^{n} (f_{2k}' + \tfrac{1}{2}f_{2k}), \tag{9}$$

$$S_n(\xi) = \tfrac{1}{2}R\sigma\, e^{-\frac{1}{2}\xi} \sum_{j=1}^{\infty} \sum_{\substack{k=0\\k\neq n}}^{\infty} \{\alpha_{jk}^{n} f_j t_k' + \beta_{jk}^{n}(f_j' + \tfrac{1}{2}f_j)t_k\}. \tag{10}$$

Here the constants† α_{jk}^n and β_{jk}^n are derived from the integrals of three associated Legendre functions and may be shown to be given by

$$\left.\begin{aligned} \alpha_{jk}^{n} &= (2n+1)\begin{pmatrix} n & j & k \\ 0 & 0 & 0 \end{pmatrix}^2, \\ \beta_{jk}^{n} &= -(2n+1)\left\{\frac{k(k+1)}{j(j+1)}\right\}^{\frac{1}{2}} \begin{pmatrix} n & j & k \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} n & j & k \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}\right\} \tag{11}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the $3-j$ symbols. A table of these quantities and the algorithm to compute them numerically has been given by Rotenberg *et al.* (1959, p. 37).

Equations (7) are a set of linear simultaneous equations which are homogeneous in the functions $t_n(\xi)$ with coefficients which can be calculated numerically from the functions $f_n(\xi)$. The theoretical problem is to solve these equations for a given R and σ subject to the boundary conditions (6). In practice, condition (6*b*) could be enforced at some finite but large value of ξ as an approximation. Similarly condition (4*b*) could be imposed at a large enough value of ξ to give an approximate boundary condition when (3) is solved numerically using two-dimensional finite differences. An improved condition in both cases may however be deduced from Oseen theory.

3. Solution for large values of ξ

The Oseen-type solution of (3) is the leading term in a solution which is valid as $\xi \to \infty$. This is obtained by putting $u = \cos\theta$ and $v = -\sin\theta$ in (3). A solution of the resulting equation which has $\partial T/\partial\theta = 0$ on $\theta = 0$ and $\theta = \pi$ is

$$T(\xi, \theta) = \exp\{\chi\cos\theta - \tfrac{1}{2}\xi\} \sum_{n=0}^{\infty} A_n K_{n+\frac{1}{2}}(\chi) P_n(\cos\theta), \tag{12}$$

† These constants differ from those given by Dennis & Walker (1971) for the steady flow problem.

where $\chi = \frac{1}{4}R\sigma\, e^{\xi}$, $K_{n+\frac{1}{2}}$ is the modified Bessel function of the second kind and the A_n are constants. Illingworth (1963) has described the classical theory, in which, if $R\sigma$ is small, a valid first approximation to the whole temperature field may be obtained by determining the constants A_n such that (4 *a*) is satisfied. If $R\sigma$ is not small, the expression (12) is valid only as $\xi \to \infty$ and the constants A_n would be determined by matching (12) with some valid inner solution. The present object is to match with an inner numerical solution and this is done by using (12) as a boundary condition for large ξ.

As $\xi \to \infty$, the Bessel functions in (12) may be replaced by the leading term in their asymptotic expansion for large ξ, so that

$$T(\xi,\theta) \sim f(\theta)\,\chi^{-1}\exp\{\chi(\cos\theta - 1)\}\dagger \tag{13}$$

as $\chi \to \infty$, where $$f(\theta) = \tfrac{1}{2}(\tfrac{1}{2}R\sigma\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n P_n(\cos\theta).$$

Thus the temperature is exponentially small everywhere except in a wake region for which $\chi(1-\cos\theta) = O(1)$, i.e. for which $\theta = O(\chi^{-\frac{1}{2}})$ as $\chi \to \infty$. Within this region we can introduce a new angular co-ordinate ω defined by

$$\omega = (\tfrac{1}{2}\chi)^{\frac{1}{2}}\,\theta \tag{14}$$

which has the effect of scaling θ with respect to the angular breadth of the wake. As $\chi \to \infty$, the physical range of the co-ordinate θ, from $\theta = 0$ to $\theta = \pi$, corresponds to the range $(0, \infty)$ for ω. It is now possible to use the properties of the wake to deduce a condition on $t_n(\xi)$ as $\xi \to \infty$.

It may be seen from (5) that

$$t_n(\xi) = \frac{2n+1}{2}\int_0^{\pi} T(\xi,\theta)\,P_n(\cos\theta)\sin\theta\, d\theta. \tag{15}$$

For large ξ, the expression (13) for $T(\xi,\theta)$ may be substituted in (15). Within the temperature wake, at large distances from the sphere, θ will be small and we may write $1-\cos\theta \sim \frac{1}{2}\theta^2$, $\sin\theta \sim \theta$ and $P_n(\cos\theta) \sim 1$.‡ The change of variable (14) now leads to

$$t_n(\xi) \sim 2D_n\, e^{-2\xi}\int_0^{\infty} \omega\, e^{-\omega^2}\, d\omega = D_n\, e^{-2\xi}\S \quad \text{as} \quad \xi \to \infty. \tag{16}$$

Values of the constants D_n are not known, but we can use (16) to match an inner numerical solution at $\xi = l$, where l is sufficiently large, to the Oseen solution by requiring that

$$t_n(l) = e^{-2h}\, t_n(l-h), \tag{17}$$

where h is the grid size used in a numerical solution.

† A referee has suggested that, in view of the possible slow convergence of $\Sigma A_n P_n(\cos\theta)$ associated with its rapid variation with respect to θ in the wake, the leading error term in (13) may be divergent for some values of θ, corresponding to a non-uniformity in the expansion. Because the A_n are not known this remains unresolved here, but it is nevertheless assumed that the leading term in the expansion for T is correctly given by (13) for all θ.

‡ For θ small and n large (of $O(1/\theta)$), the leading term in $P_n(\cos\theta)$ is of $O(1)$.

§ This result can also be obtained without introducing ω by writing the exponential in (12) as a series of modified Bessel functions $I_{n+\frac{1}{2}}(\chi)$ and $P_n(\cos\theta)$ and subsequently taking the leading term in (15) for large ξ.

Equation (17) gives a boundary condition to be applied at $\xi = l$ when solving the set of equations (7). A condition to be applied at a large enough value $\xi = l$ when (3) is solved using two-dimensional finite differences is obtained by elimination of the unknown function $f(\theta)$ from (13) using the values of $T(\xi, \theta)$ on the grid lines $\xi = l-h$ and $\xi = l$. This gives

$$T(l, \theta) = T(l-h, \theta) \exp\{(\chi_l - \chi_{l-h})(\cos\theta - 1) - h\}, \tag{18}$$

where χ_l denotes the value of χ at $\xi = l$. This condition is similar to that given by Dennis, Hudson & Smith (1968) in the case of heat transfer from a circular cylinder.

4. Calculation procedure

In solving the set of equations (7), a truncation of order n_0 was defined by setting all $t_n(\xi)$ for $n > n_0$ equal to zero. The finite set of n_0 equations was then solved numerically. There is no basic way of choosing n_0 to give a good approximation to the temperature field for a given R and σ and this problem can only be overcome by increasing n_0 until no appreciable change in the solution occurs. A finite-difference method of solution was adopted in which all derivatives were approximated by three-point central differences. Thus a tridiagonal matrix problem was obtained for each $t_n(\xi)$ and was solved by the direct method described by Rosser (1967). The functions $t_n(\xi)$ satisfied condition (6a) on the sphere and condition (17) on the outer boundary $\xi = l$. The functions $f_n(\xi)$ were obtained from the solutions of Dennis & Walker (1971).

The truncated set of equations (7) was solved by a general iterative procedure. A sequence of iterates

$$\{t_n^{(j)}(\xi)\} \quad (n = 0, 1, 2, \ldots, n_0; \quad j = 1, 2, 3, \ldots)$$

was obtained in the following manner. Suppose that the iterate with superscript (j) has been completed. Equations (7) were then solved sequentially from $n = 0$ to n_0, always using the most recently available information to calculate the functions $S_n(\xi)$ defined by (10). As each function was obtained, denoted by $\bar{t}_n(\xi)$, the components of the iterate with superscript $(j+1)$ were defined by

$$t_n^{(j+1)}(\xi) = \epsilon \bar{t}_n(\xi) + (1-\epsilon)\, t_n^{(j)}(\xi),$$

where ϵ is an empirical parameter. In general as R increases the parameter ϵ must be reduced or divergence of the scheme will result. For $R \leqslant 1$, $\epsilon = 0{\cdot}05$ was used but this had to be reduced to $0{\cdot}01$ at the highest value of R considered ($R = 10$). The values however are not necessarily the largest ones compatible with convergence.

The iterative scheme was continued until ultimately convergence was attained, which was decided by the test

$$|t_n^{(j+1)}(\xi) - t_n^{(j)}(\xi)| < 10^{-4} \quad \text{for all } \xi \text{ and for all } n \leqslant n_0.$$

The number n_0 of terms retained in the series varied from a minimum of $n_0 = 7$ at the lowest Reynolds numbers to a maximum of $n_0 = 17$ at the highest ($R = 10$).

The flow field solutions of Dennis & Walker (1971) were obtained using a grid size of $h = 0{\cdot}0245$ and with the outer boundary at $l = 4{\cdot}9$ (corresponding to a dimensionless radial distance of $r/a = 134{\cdot}2$ from the centre of the sphere) for $R \leqslant 10$. These same values of the parameters were used in the present solutions for $\sigma = 0{\cdot}73$ and $\sigma = 1$. For the case $\sigma = 8$, the parameters l and h were halved and the velocity field at the intermediate grid points obtained by a five-point Lagrangian interpolation formula. The logic behind this step is based on the notion of a thermal boundary layer having thickness of order $\sigma^{-\frac{1}{3}}$.

It becomes increasingly difficult to obtain reliable solutions as R and σ increase, and, further, the Reynolds-number range for reliable solutions decreases as σ increases. To obtain at least some check on the solutions obtained by the series truncation method, a set of solutions for $R = 0{\cdot}5$–20 was obtained in the case $\sigma = 0{\cdot}73$ by an independent method and using an independent velocity field. The velocity field was obtained by solving the Navier–Stokes equations, expressed in terms of the usual two simultaneous equations for the stream function and vorticity, by two-dimensional finite differences. The details will not be described here except to say that basically the same formulation, using central differences, as that given by Jenson (1959) was adopted with, however, an improvement in the treatment of the boundary conditions far from the sphere. An equation similar to (18), derived from Oseen theory, was used as a boundary condition for the vorticity on the far boundary $\xi = l$. An equation for the stream function, again obtained by consideration of the nature of the flow in the wake, was used as a boundary condition for ψ on $\xi = l$.

The flow field was calculated using this method for $R = 0{\cdot}5$, 1, 5, 10 and 20. In each case a grid size $h = 0{\cdot}1$ in the ξ direction with a grid size of $\frac{1}{30}\pi$ in the θ direction was used. The value of l was taken as 3. The effect on the solutions of varying the parameter l was studied and it was found to be well within a 1% tolerance. The computed velocity fields were used in (3) for the same range of Reynolds numbers and with $\sigma = 0{\cdot}73$. Numerical solutions of (3) were found using conditions (4a), condition (18) and the conditions of symmetry about $\theta = 0$ and $\theta = \pi$ as boundary conditions. The same grid sizes were used for the flow field calculations and the same value of l was used. All derivatives in (3) were approximated by central differences and the difference equations were solved by the method of successive over-relaxation.

5. Calculated results

The local amount of heat transferred per unit area per unit time from the sphere to the fluid is

$$q(\theta) = -\kappa \frac{T_s - T_\infty}{a}\left(\frac{\partial T}{\partial \xi}\right)_{\xi=0},$$

where κ is the thermal conductivity. The local Nusselt number is defined as

$$N(\theta) = \frac{2aq(\theta)}{\kappa(T_s - T_\infty)} = -2\left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} = -2\sum_{n=0}^{\infty} t_n'(0)\,P_n(\cos\theta) \tag{19}$$

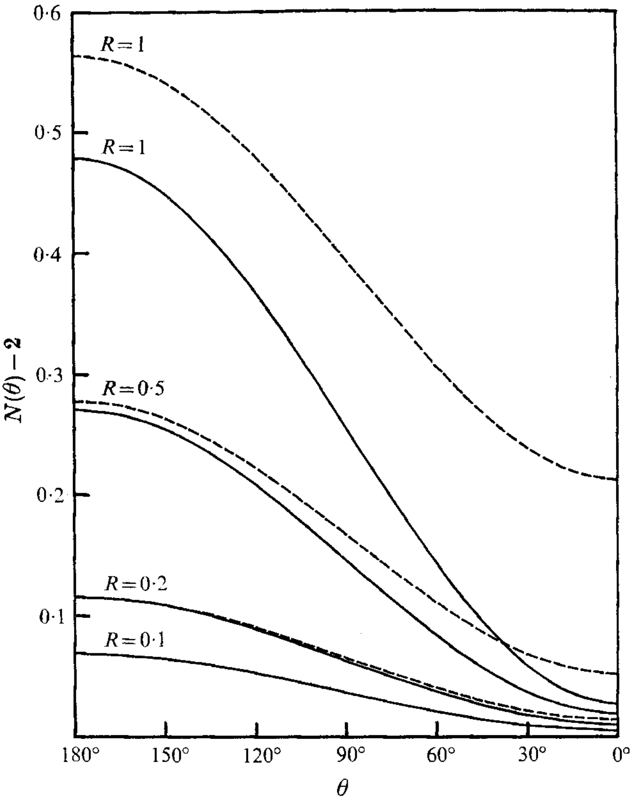

FIGURE 1. Local Nusselt number. —, present results; – –, Rimmer (1968, 1969).

and the mean Nusselt number as

$$N_m = -\int_0^\pi \left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} \sin\theta\, d\theta = -2t'(0). \tag{20}$$

Two comparisons of the present results are of interest. First, there is the result given by Rimmer (1968):

$$N_m \sim 2 + \tfrac{1}{2}\sigma R + \tfrac{1}{4}\sigma^2 R^2 \log\left(\tfrac{1}{2}\sigma R\right) + \tfrac{1}{4} f(\sigma)\, \sigma^2 R^2, \tag{21}$$

where $$f(\sigma) = \tfrac{1}{4}\left[(2\sigma^2 - \sigma + 4\gamma - \tfrac{173}{40}) + 2(\sigma^3 - 3\sigma - 2)\log\sigma - 2(\sigma+1)^2(\sigma-2)\log(\sigma+1)\right].$$

This result is based on the method of matched inner and outer expansions. Here, $f(\sigma)$ is the corrected expression given by Rimmer (1969) and γ is Euler's constant. It may be shown from Rimmer's results that the local Nusselt number $N(\theta)$ is given by

$$N(\theta) \sim N_m - \tfrac{3}{8}R\sigma\left\{1 + \frac{3R}{16}(1-4\sigma)\right\} P_1(\cos\theta) + \tfrac{1}{128}R^2\sigma\left\{\tfrac{13}{5} - \tfrac{33}{7}\sigma\right\} P_2(\cos\theta). \tag{22}$$

In figure 1, the present results obtained by the series truncation method for the local Nusselt number $N(\theta)$ for $\sigma = 0{\cdot}73$ and $R = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}5$ and 1 are com-

R	$N(0)$	$N(\pi)$	N_m
0·1	2·006	2·068	2·037
0·2	2·011	2·116	2·064
0·5	2·026	2·272	2·151
1	2·034	2·482	2·260
5	2·038	3·656	2·857
10	2·045	4·662	3·358
20	2·062	6·136	4·065

TABLE 1. Calculated heat-transfer coefficients for $\sigma = 0{\cdot}73$.

R	$N(0)$	$N(\pi)$	N_m
0·1	2·007	2·060	2·034
0·2	2·015	2·116	2·066
0·5	2·053	2·278	2·166
1·0	2·213	2·564	2·393

TABLE 2. Theoretical heat-transfer coefficients due to Rimmer (1968, 1969).

pared with Rimmer's (1968, 1969) theory (equation (22)). At $R = 0{\cdot}1$ the results are almost identical. As R increases there is a progressive deviation from Rimmer's theory, particularly near the back of the sphere. At $R = 0{\cdot}5$, the agreement is still within about 1·5 % over the entire equator of the sphere but at $R = 1$ the theory tends to become inadequate. Similar conclusions were obtained for $\sigma = 1$, the discrepancy at $R = 0{\cdot}5$ in this case being about 3·4 %. For a given value of R the discrepancy tends to increase as σ increases.

The results obtained for the case $\sigma = 0{\cdot}73$ using the method of two-dimensional finite differences were found, where comparison is possible, to be in very good agreement with those calculated using the method of series truncation. On the whole the agreement of the calculated Nusselt numbers was found to be well within 1 % for the two methods, which are completely independent. Some calculated results for $\sigma = 0{\cdot}73$ are shown in table 1. These values were derived mainly from the results of the two-dimensional finite-difference method, with the exception of the values at $R = 0{\cdot}1$ and 0·2, for which only the series truncation method was employed. Some theoretical values obtained from the expression of Rimmer (1968) are shown in table 2 for comparison.

The second point of interest is a comparison of the present results with experiment. Kramers (1946) carried out experiments with various flowing media and found that the formula

$$N_m = 2 + 1{\cdot}3\sigma^{0{\cdot}15} + 0{\cdot}66\sigma^{0{\cdot}31}R^{0{\cdot}50} \tag{23}$$

described his results. Since this formula does not reduce to the Stokes result $N_m = 2$ as $R \to 0$, it is not valid for small R. A later extensive experimental investigation by Ranz & Marshall (1952) indicated a correlation

$$N_m = 2 + 0{\cdot}60R^{\frac{1}{2}}\sigma^{\frac{1}{3}}. \tag{24}$$

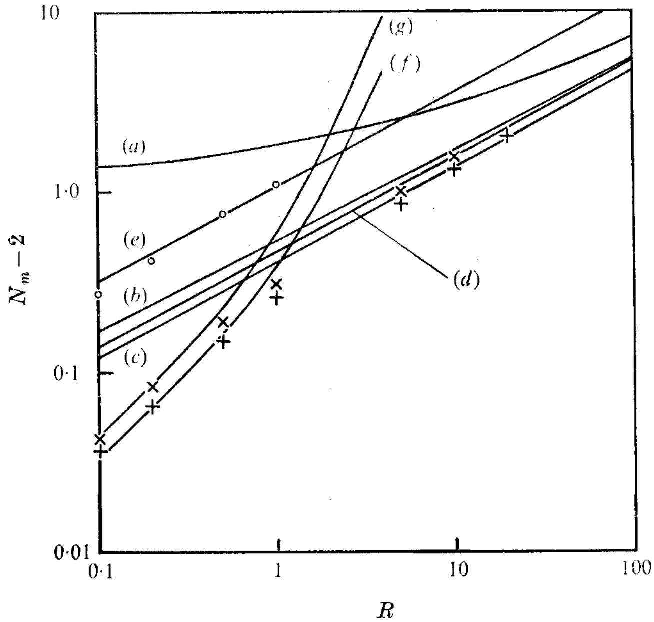

FIGURE 2. Comparison of mean Nusselt number with theory and experiment. (i) Present results: +, $\sigma = 0{\cdot}73$; ×, $\sigma = 1$; ○, $\sigma = 8$. (ii) Experimental correlations: (a) equation (23), $\sigma = 0{\cdot}73$, (b) equation (24), $\sigma = 0{\cdot}73$, (c) equation (25), $\sigma = 0{\cdot}73$, (d) equation (25), $\sigma = 1$, (e) equation (25), $\sigma = 8$. (iii) Theory (Rimmer 1968, 1969): (f) $\sigma = 0{\cdot}73$, (g) $\sigma = 1$.

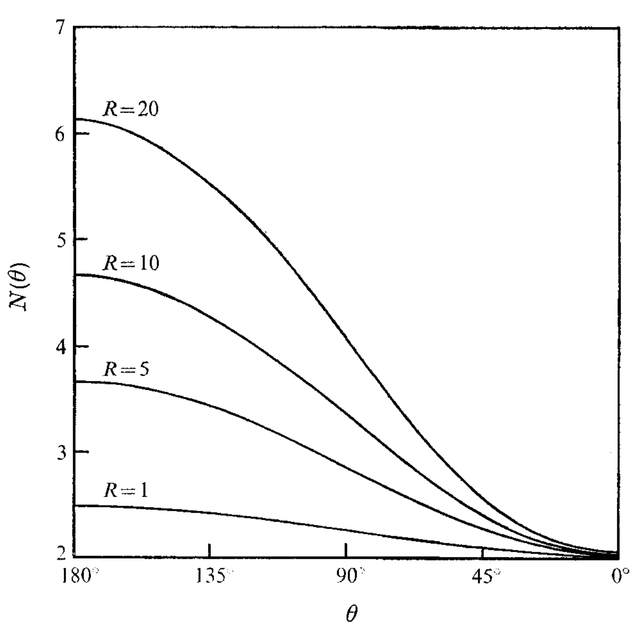

FIGURE 3. Local heat transfer for $\sigma = 0{\cdot}73$.

The most recent formula is that given by Whitaker (1972), which is

$$N_m = 2+(0\cdot 4R^{\frac{1}{2}}+0\cdot 06R^{\frac{2}{3}})\,\sigma^{0\cdot 4}. \tag{25}$$

Equation (25) is based on a correlation of the experimental data of Vliet & Leppert (1961) for water, Kramers (1946) for air, water and oil and Yuge (1960) for air as well as a theoretical argument due to Richardson (1968). The present results for the mean Nusselt number are plotted for Prandtl numbers of $\sigma = 0\cdot 73$, 1 and 8 in figure 2. The results are compared here with the experimental formulae (23), (24) and (25) as well as the theory of Rimmer (1968, 1969).

It is the conclusion of the present study that the theory of Rimmer (1968) represents N_m best for $\sigma \sim O(1)$ and R small. However, the theory becomes inadequate with increasing R and the formula (25) then represents N_m best. Equation (25) appears to fit the present results better than either (23) or (24), at least for the Prandtl numbers considered. Finally, the variation of $N(\theta)$ is shown over the equator of the sphere for $\sigma = 0\cdot 73$ and the range $R = 1$–20 in figure 3. The behaviour of $N(\theta)$ with increasing R tends to be in agreement with the model proposed by Acrivos *et al.* (1965) for a circular cylinder in that the variation of $R^{-\frac{1}{2}}N(\pi)$ is tending to become slower as R increases, in accordance with boundary-layer theory, whereas there is only a slight upward tendency of $N(0)$. According to the model this latter quantity should become roughly constant.

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